Surface integrals of scalar fields wrt surface area.

Recall that $d\Phi$ of a (usually tiny) vector $\Phi$ is just the (positive) length of $\Phi$, e.g. $d\Phi(\frac{1}{3}) = \sqrt{14}$. Similarly, we define $d\$ of an ordered parallelogram $\$:

area of $\$, i.e. $d\$(a,b) = \|a \times b\|$

Recall that $\int f d\Phi$ is defined by a (Riemann) sum, where $f$ takes in points $x_i$ and $d\Phi$ takes in vectors $\Phi_i$ (tiny bits of core; direction doesn't affect $d\Phi(\Phi_i) = \|\Phi_i\|$!), and we add up the terms $f(x_i) d\Phi(\Phi_i)$.

Similarly, $\int \int f d\$ is defined by a (Riemann) sum, where $f$ takes in points $x_i$, and $d\$ takes in parallelograms $\$, (tiny bits of surface; order/direction doesn't affect $d\$(a,b) = \|a \times b\|$!), and we add up the terms $f(x_i) d\$(a,b)$.
You can just think of $\iiint_S f \, d\gamma$ like a "sum of outputs of $f$ from inputs on $S$," which is good for most scientific applications, but if you insist on thinking geometrically, it is the (signed) volume between $S$ and $\text{graph}(f) \subseteq \mathbb{R}^3 \times \mathbb{R}^1 = \mathbb{R}^4$.

To compute, we usually parametrize $S$ once, say $\Phi : D \rightarrow S$ with $D$ in $\mathbb{R}^2$ and $S$ in $\mathbb{R}^3_{x,y,z}$, and think of $S$ being made up of the tiny parallelograms \( \|ru \times rv\| \) (from the LAF!),

Since $d\gamma (ru,rv) = \|ru \times rv\| = \|ru \times rv\| dudv$,

we get the formulas

\[
\iiint_S f \, d\gamma = \iiint_D f(u,v) \|ru \times rv\| \, dudv
\]
Please see Stewart 16.7 for examples!

Remarks: 1) The (Riemann) sum definition of $\int_S f \, dS$ does not involve parametrization! So the answer doesn't depend on any parametrization of $S$.

2) Recall that $dA$ had a "stand alone" definition $dA = \sqrt{(dx)^2 + (dy)^2}$. So does $dS$, but it involves a "wedge product" operation:

$$dS = \sqrt{(dx\wedge dy)^2 + (dy\wedge dz)^2 + (dz\wedge dx)^2}$$

Which is easy to understand (with the right explanation!), but it's not part of this course.